NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Standard 1 - Worthwhile Mathematical Tasks

The teacher of mathematics should pose tasks that are based on-

- sound and significant mathematics;
- knowledge of students' understandings, interests, and experiences;
- knowledge of the range of ways that diverse students learn mathematics;

and that

- engage students' intellect;
- develop students' mathematical understandings and skills;
- stimulate students to make connections and develop a coherent framework for mathematical ideas;
- call for problem formulation, problem solving, and mathematical reasoning;
- promote communication about mathematics;
- represent mathematics as an ongoing human activity;
- display sensitivity to, and draw on, students' diverse background experiences and dispositions;
- promote the development of all students' dispositions to do mathematics.

Elaboration

Teachers are responsible for the quality of the mathematical tasks in which students engage. A wide range of materials exists for teaching mathematics: problem booklets, computer software, practice sheets, puzzles, manipulative materials, calculators, textbooks, and so on. These materials contain tasks from which teachers can choose. Also, teachers often create their own tasks for students: projects, problems, worksheets, and the like. Some tasks grow out of students' conjectures or questions. Teachers should choose and develop tasks that are likely to promote the development of students' understandings of concepts and procedures in a way that also fosters their ability to solve problems and to reason and communicate mathematically. Good tasks are ones that do not separate mathematical thinking from mathematical concepts or skills, that capture students' curiosity, and that invite them to speculate and to pursue their hunches. Many such tasks can be approached in more than one interesting and legitimate way; some have more than one reasonable solution. These tasks, consequently, facilitate significant classroom discourse, for they require that students reason about different strategies and outcomes, weigh the pros and cons of alternatives, and pursue particular paths.

In selecting, adapting, or generating mathematical tasks, teachers must base their decisions on three areas of concern: the mathematical content, the students, and the ways in which students learn mathematics.

In considering the mathematical content of a task, teachers should consider how appropriately the task represents the concepts and procedures entailed. For example, if students are to gather, summarize, and interpret data, are the statistics they are expected to generate appropriate? Does it make sense to calculate a mean? If there is an explanation of a procedure, such as calculating a mean, does that explanation focus on the underlying concepts or is it merely mechanical? Teachers must also use a curricular perspective, considering the potential of a task to help students progress in their cumulative understanding of a particular domain and to make connections among ideas they have studied in the past and those they will encounter in the future.

A second content consideration is to assess what the task conveys about what is entailed in doing mathematics. Some tasks, although they deal nicely with the concepts and procedures, involve students in simply producing right answers. Others require students to speculate, to pursue alternatives, to face decisions about whether or not their approaches are valid. For example, one task might require students to find means, medians, and modes for given sets of data. Another might require them to decide whether to calculate means, medians, or modes as the best measures of central tendency, given particular sets of data and particular claims they would like to make about the data, then to calculate those statistics, and finally to explain and defend their decisions. Like the first task, the second would offer students the opportunity to practice finding means, medians, and modes. Only the second, however, conveys the important point that summarizing data involves decisions related to the data and the purposes for which the analysis is being used. Tasks should foster students' sense that mathematics is a changing and evolving domain, one in which ideas grow and develop over time and to which many cultural groups have contributed. Drawing on the history of mathematics can help teachers to portray this idea: exploring alternative numeration systems or investigating non-Euclidean geometries, for example. Fractions evolved out of the Egyptians' attempts to divide quantities four things shared among ten people. This fact could provide the explicit basis for a teacher's approach to introducing fractions.

A third content consideration centers on the development of appropriate skill and automaticity. Teachers must assess the extent to which skills play a role in the context of particular mathematical topics. A goal is to create contexts that foster skill development even as students engage in problem solving and reasoning. For example, elementary school students should develop rapid facility with addition and multiplication combinations. Rolling pairs of dice as part of an investigation of probability can simultaneously provide students with practice with addition. Trying to figure out how many ways 36 desks can be arranged in equal-sized groups - and whether there are more or fewer possible groupings with 36, 37, 38, 39, or 40 desks - presses students to produce each number's factors quickly. As they work on this problem, students have concurrent opportunities to practice multiplication facts and to develop a sense of what factors are. Further, the problem may provoke interesting questions: How many factors does a number have? Do larger numbers necessarily have more factors? Is there a number that has more factors than 36? Even as students pursue such questions, they practice and use multiplication facts, for skill plays a role in problem solving at all levels. Teachers of algebra and geometry must similarly consider which skills are essential and why and seek ways to develop essential skills in the contexts in which they matter. What do students need to memorize? How can that be facilitated?

The content is unquestionably a crucial consideration in appraising the value of a particular task. Defensible reasoning about the mathematics of a task must be based on a thoughtful understanding of the topic at hand as well as of the goals and purposes of carrying out particular mathematical processes.

Teachers must also consider the students in deciding on the appropriateness of a given task. They must consider what they know about their particular students as well as what they know more generally about students from psychological, cultural, sociological, and political perspectives. For example, teachers should consider gender issues in selecting tasks, deliberating about ways in which the tasks may be an advantage either to boys or to girls - and a disadvantage to the others - in some systematic way.

In thinking about their particular students, teachers must weigh several factors. One centers on what their students already know and can do, what they need to work on, and how much they seem ready to stretch intellectually. Well-chosen tasks afford teachers opportunities to learn about their students' understandings even as the tasks also press the students forward. Another factor is their students' interests, dispositions, and experiences. Teachers should aim for tasks that are likely to engage their students' interests. Sometimes this means choosing familiar application contexts: for example, having students explore issues related to the finances of a school store or something in the students' community. Not always, however, should concern for "interest" limit the teacher to tasks that relate to the familiar everyday worlds of the students; theoretical or fanciful tasks that challenge students intellectually are also interesting: number theory problems, for instance. When teachers work with groups of students for whom the notion of "argument" is uncomfortable or at variance with community norms of interaction, teachers must consider carefully the ways in which they help students to engage in mathematical discourse. Defensible reasoning about students must be based on the assumption that all students can learn and do mathematics, that each one is worthy of being challenged intellectually. Sensitivity to the diversity of students' backgrounds and experiences is crucial in selecting worthwhile tasks.

Knowledge about ways in which students learn mathematics is a third basis for appraising tasks. The mode of activity, the kind of thinking required, and the way in which students are led to explore the particular content all contribute to the kind of learning opportunity afforded by the task. Knowing that students need opportunities to model concepts concretely and pictorially, for example, might lead a teacher to select a task that involves such representations. An awareness of common student confusions or misconceptions around a certain mathematical topic would help a teacher to select tasks that engage students in exploring critical ideas that often underlie those confusions. Understanding that writing about one's ideas helps to clarify and develop one's understandings would make a task that requires students to write explanations look attractive. Teachers' understandings about how students learn mathematics should be informed by research as well as their own experience. Just as teachers can learn more about students' understandings from the tasks they provide students, so, too, can they gain insights into how students learn mathematics. To capitalize on the opportunity, teachers should deliberately select tasks that provide them with windows on students' thinking.

Vignettes

The teacher analyzes the content and how to approach it, and she considers how it connects with other mathematical ideas.

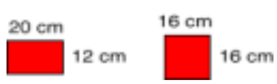
1.1 Mrs. Jackson is thinking about how to help her students learn about perimeter and area. She realizes that learning about perimeter and area entails developing concepts, procedures, and skills. Students need to understand that the perimeter is the distance around a region and the area is the amount of space inside the region and that length and area are two fundamentally different kinds of measure. They need to realize that perimeter and area are not directly related - that, for instance, two figures can have the same perimeter but different areas. Students also need to be able to figure out the perimeter and the area of a given region. At the same time, they should relate these to other measures with which they are familiar, such as measures of volume or weight.

Mrs. Jackson examines two tasks designed to help upper elementary-grade students learn about perimeter and area. She wants to compare what each has to offer.

Task 1 requires little more than remembering what "perimeter" and "area" refer to and the formulas for calculating each. Nothing about thin task requires students to ponder the relationship between perimeter and area. This task is not likely to engage students intellectually; it does not entail reasoning or problem solving.

TASK 1:

Find the area and perimeter of each rectangle:



Task 2 can engage students intellectually because it challenges them to search for something. Although accessible to even young students, the problem is not immediately solvable. Neither is it clear how best to approach it. A question that students confront as they work on the problem is how to determine that they have indeed found the largest or the smallest play space. Being able to justify an answer and to show that a problem is solved are critical components of mathematical reasoning and problem solving. The problem yields to a variety of tools - drawings on graph paper, constructions with rulers or compasses, tables, calculators - and lets students develop their understandings of the concept of area and its relationship to perimeter. They can investigate the patterns that emerge in the dimensions and the relationship between those dimensions and the area. This problem may also prompt the question of what "largest" or "smallest," "most" or "least" mean, setting the stage for making connections in other measurement contexts.

TASK 2:

Suppose you had 64 meters of fence with which you were going to build a pen for your large dog, Bones. What are some different pens you can make if you use all the fencing? What is the pen with the least play space? What is the biggest pen you can make - the one that allows Bones the most play space? Which would be best for running?

1.2 Ms. Pierce is a first-year teacher in a large middle school. She uses a mathematics textbook, published about ten years ago, that her department requires her to follow closely. In the middle of a unit on fractions with her seventh graders, Ms. Pierce is examining her textbook's treatment of division with fractions. She is trying to decide what its strengths and weaknesses are and whether and how she should use it to help her students understand division with

> fractions.

Many beginning and experienced teachers are in the same position as this teacher: having to follow a textbook quite closely. Appraising and deciding how to use textbook material is critical.

> She notices that the textbook's emphasis is on the mechanics of carrying out the procedure ("dividing by a number is the same as multiplying by its reciprocal"). The text tells students that they "can use reciprocals to help" them divide by fractions and gives them a few examples of the procedure.

The teacher wants her students to understand what it means to divide by a fraction, not just learn the mechanics of the procedure.

> The picture at the top of one of the pages shows some beads of a necklace lined up next to a ruler - an attempt to represent, for example, that there are twenty-four 3/4-inch beads and forty-eight 3/8-inch beads in an eighteen-inch necklace. Ms. Pierce sees that this does represent what it means to divide by 3/4 or by 3/8 - that the question is, "How many three-fourths or three-eighths are there in eighteen?" Still, when she considers what would help her students understand this, she does not think that this representation is adequate. She also suspects that students may not take this section seriously, for they tend to believe that mathematics means memorizing rules rather than understanding why the rules work.

The teacher senses that the idea of "using the reciprocal" is introduced almost as a trick, lacking any real rationale or connection to the pictures of necklaces. Furthermore, division with fractions seems to be presented as a new topic, unconnected to anything the students might already know, such as division of whole numbers.

The practice exercises involve dividing one fraction by another, and the "problems" at the end do not involve reasoning or problem solving.

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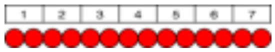
> Ms. Pierce is concerned that these pages are likely to reinforce that impression. She doesn't see anything in the task that would emphasize the value of understanding why, nor that would promote mathematical discourse.

The teacher considers what she knows about her students - what they know and what is likely to interest them.

> Thinking about her students, Ms. Pierce judges that these two pages require computational skills that most of her students do have (i.e., being able to produce the reciprocal of a number, being able to multiply fractions) but that the exercises on the pages would not be interesting to them. Nothing here would engage their thinking.

The model used is a linear one rather than the pie or pizza diagrams most often used to represent fractions. The teacher sees the need for students to develop varied representations. Also, different representations make sense to different students. The teacher wants the task to help students make connections - in this example, between multiplication and division and between division of whole numbers and division of fractions.

> Looking at the pictures of the necklaces gives Ms. Pierce an idea. She decides that she can use this idea, so she copies the drawing only. She will include at least one picture with beads of some whole number length - 2-inch beads, for example. She will ask students to examine the pictures and try to write some kind of number sentence that represents what they see. For example, this 7-inch bracelet has 14 half-inch beads:



This could be represented as $7 \div \frac{1}{2}$ or 7×2. She will try to help them to think about the reciprocal relationship between multiplication and division and the meaning of dividing something by a fraction or by a whole number. Then, she thinks, she could use some of the exercises on the second page but, instead of just having the students compute the answers, she will ask them, in pairs, to write stories for each of about five exercises.

Writing stories to go with the division sentences may help students to focus on the meaning of the procedure.

The teacher keeps her eye on the bigger curricular picture as she selects and adapts tasks. Juxtaposing whole number and fraction division will help her students review division and make connections.

She decides she will also provide a couple of other examples that involve whole number divisors: $28 \div 8$ and $80 \div 16$, for example.

Ms. Pierce feels encouraged from her experience with planning this lesson and thinks that revising other textbook lessons will be feasible. Despite the fact that she is supposed to be following the text closely, Ms. Pierce now thinks that she will be able to adapt the text in ways that will significantly improve what she can do with her students this year.

1.3 After recently completing a unit on multiplication and division, a fourth-grade class has just begun to learn about factors and multiples. Their teacher is using the calculator as a tool for this topic. This approach is new for her. The school has just purchased for the first time a set of calculators, which all the classrooms share. She and many of her colleagues attended a workshop recently on different uses of calculators.

Using the automatic constant feature of their calculators (that is, that pressing 5 + = = = ... yields 5, 10, 15, 20,....on the display), the fourth graders have generated lists of the multiples of different numbers. They have also used the calculator to explore the factors of different numbers. To encourage the students to deepen their understanding of numbers, the teacher has urged them to look for patterns and to make conjectures. She asked them, "Do you see any patterns in the lists you are making? Can you make any guesses about any of those patterns?"

The teacher uses this exploratory task to spur students' mathematical thinking. She knows that the initial task is likely to generate further, more focused tasks based on the students' conjectures. The calculators help the students in looking for patterns.

Two students have raised a question that has attracted the interest of the whole class:

> Are there more multiples of 3 or more multiples of 8?

All year, this teacher has encouraged her students to take intellectual risks by asking questions.

Judging that this question is a fruitful one, the teacher picks up on the students' idea and uses it to further the direction of the class's exploration, even bringing up questions about infinity.

The teacher encourages them to pursue the question, for she sees that this question can engage them in the concept of multiples as well as provide a fruitful context for making mathematical arguments. She realizes that the question holds rich mathematical potential and even brings up questions about infinity. "What do the rest of you think?" she asks. "How could you investigate this question? Go ahead and work on this a bit on your own or with a partner and then let's discuss what you come up with."

The children pursue the question excitedly. The calculators are useful once more as they generate lists of the multiples of 3 and the multiples of 8. Groups are forming around particular arguments. One group of children argues that there are more multiples of 3 because in the interval between 0 and 20 there are more multiples of 3 than multiples of 8. Another group is convinced that the multiples of 3 are "just as many as the multiples of 8 because they go on forever." A few children, thinking there should be more multiples of 8 because 8 is greater than 3, form a new conjecture about numbers - that the larger the number, the more *factors* it has.

The question promotes mathematical reasoning, eliciting at least three competing and, to fourth graders, compelling mathematical arguments. Students are actively engaged in trying to persuade other members of the class of the validity of their argument.

> The teacher is pleased with the ways in which opportunities for mathematical reasoning are growing out of the initial exploration. She likes the way in which they are making connections between multiples and factors. She also notes that students already seem quite fluent using the terms *multiple* and *factor*.

The task has stimulated students to formulate a new problem. The idea that lessons can raise questions for students to pursue is part of an emphasis on mathematical inquiry.

The teacher provides a context far dealing with students' conjectures. She is also able to formulate tasks out of the students' ideas and questions when it seems fruitful.

> Although it is nearing the end of class, the teacher invites them to present to the rest of the class their conjecture that the larger the number, the more factors it has. She suggests that the students record it in their notebooks and discuss it in class tomorrow. Pausing for a moment before she sends them out to recess, she decides to provoke their thinking a little and remarks, "That's an interesting conjecture. Let's just think about it for a sec. How many factors does, say, 3 have?"
>
> "Two," call out several students.
>
> "What are they?" she probes. "Yes, Deng?"

The teacher provides practice in multiplication facts at the same time that she engages the students in considering their peers' conjecture.

> "1 and 3," replies Deng quickly.
>
> "Let's try another one," continues the teacher. "What about 20?"
>
> After a moment, several hands shoot up. She pauses to allow students to think and asks, "Natasha?"
>
> "Six- 1 and 20, 2 and 10, 4 and 5," answers Natasha with confidence.

The teacher does not want to give them a key to challenging the conjecture, but she does want to get them into investigating it.

She tries to spur them on to pursuing this idea on their own.

> The teacher throws out a couple more numbers - 9 and 15. She is conscious of trying to use only numbers that fit the conjecture. With satisfaction, she notes that most of the students are quickly able to produce all the factors for the numbers she gives them. Some used paper and pencil, some used calculators, and some did a combination of both. As she looks up at the clock, one child asks, "But what about 17? It doesn't seem to work."

The teacher deliberately leaves the question unanswered. She wants to encourage them to persevere and not expect her to give the answers.

> "That's one of the things that you could examine for tomorrow. I want all of you to see if you can find out if this conjecture always holds."
>
> "I don't think it'll work for odd numbers," says one child.
>
> "Check into it," smiles the teacher. "We'll discuss it tomorrow."

Summary: Tasks

The teacher is responsible for shaping and directing students' activities so that they have opportunities to engage meaningfully in mathematics. Textbooks can be useful resources for teachers, but teachers must also be free to adapt or depart from texts if students' ideas and conjectures are to help shape teachers' navigation of the content. The tasks in which students engage must encourage them to reason about mathematical ideas, to make connections, and to formulate, grapple with, and solve problems. Students also need skills. Good tasks nest skill development in the context of problem solving. In practice, students' actual opportunities for learning depend on the kind of *discourse* that the teacher orchestrates, an issue we examine in the next section.

Tasks Discourse